Subspaces of $\mathbb{R}^n$

Let W be some <u>subset</u> of $\mathbb{R}^n$, so W is a collection of (column) vectors.

<u>DEF</u> W is a <u>subspace</u> of $\mathbb{R}^n$ if 3 things are true:

(0) The $\underline{0}$-vector is in W.

(CA) ("closed under addition") If $\underline{u}$ and $\underline{v}$ are <u>any</u> two elements of W, then the <u>linear combination</u> $a\underline{u} + b\underline{v}$ is also in W

(Succinctly: $\underline{u}, \underline{v} \in W \Rightarrow \underline{u} + \underline{v} \in W$)

(CS) ("closed under scalar multiplication") If $\underline{u}$ is <u>any</u> element of W, then any scaled multiple $k\underline{u}$ is still in W. (Succinctly: $\underline{u} \in W \Rightarrow k\underline{u} \in W$ for any $k \in \mathbb{R}$)

- Technically, the entire set $\mathbb{R}^n$ is a subspace of itself, since (0), (CA), (CS) are all true for $\mathbb{R}^n$ as a whole.
  - To distinguish that technicality, sometimes we may speak of "<u>proper</u> subspaces", which are subspaces of $\mathbb{R}^n$ and are smaller than "the entire $\mathbb{R}^n$".

- Let's work out several examples of verifying the 3 subspace conditions.

(Ex 1): Within $V = \mathbb{R}^2 = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x, y \in \mathbb{R} \right\}$, let W be the set of all vectors which are scalar multiples of the vector $\underline{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

That is, $W = \{ c\underline{x} \quad : c \in \mathbb{R} \}$.

Is W a subspace of $\mathbb{R}^2$?

✓ 1) $\underline{0_2} \in W$? Yes, $\underline{0}_2$ is in W b/c $\begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0 \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}$
$= 0\underline{u}$

✓ 2) Closed under addition?

If $\underline{u}$ and $\underline{v}$ are generic vectors from W, then by definition of "being in W," it must be that $\underline{u} = a\underline{x}$ and $\underline{v} = b\underline{x}$, for some scalars a and b.

Is $\underline{u} + \underline{v}$ still a multiple of $\underline{x}$? (i.e. is $\underline{u} + \underline{v}$ still in W?)

Yes, b/c $\underline{u} + \underline{v} = a\underline{x} + b\underline{x} = (a+b)\underline{x}$

✓ 3) Closed under scalar multip.?

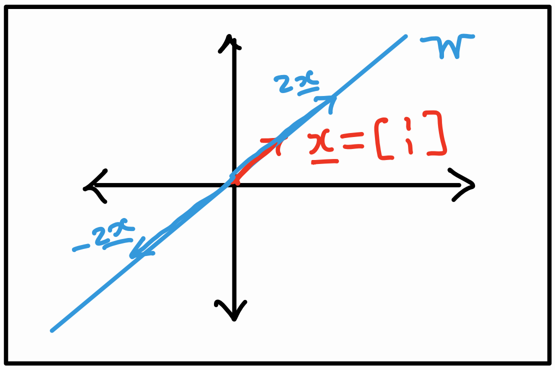
If $\underline{u}$ is a generic vector from W, then (by defin. of "being in W") $\underline{u} = a\underline{x}$ for some a.

Then if b is any other scalar,

$$b\underline{u} = b(a\underline{x}) = (ba)\underline{x},$$

which is clearly still a multiple of $\underline{x}$, so $b\underline{u} \in W$ still. Thus, yes, W closed under scalar multiplication

Since all (3) are checked, W is a subspace of $\mathbb{R}^2$.



$\mathcal{V} = \mathbb{R}^2$

(Ex 2) Within $\mathbb{R}^2$, is the set of vectors $\begin{bmatrix}x\\y\end{bmatrix}$ having $y=1$ a subspace? That is, is

$$\mathcal{W} = \left\{ \begin{bmatrix}x\\y\end{bmatrix} : x \in \mathbb{R},\ y=1 \right\}$$

$$\left(\text{alt: } \mathcal{W} = \left\{ \begin{bmatrix}x\\1\end{bmatrix} : x \in \mathbb{R} \right\} \right)$$

a subspace of $\mathbb{R}^2$?

× 1) $\underline{\underline{0} \in \mathcal{W}}$? No, b/c $\underline{0} = \begin{bmatrix}0\\0\end{bmatrix}$ does not fit the condition $y=1$.

× 2) Closed under addition?

Taking generic $\underline{u}$ and $\underline{v}$ from $\mathcal{W}$, they must be

$$\underline{u} = \begin{bmatrix}a\\1\end{bmatrix}, \quad \underline{v} = \begin{bmatrix}b\\1\end{bmatrix} \quad (\text{for } \underline{\text{some}}\ a, b)$$

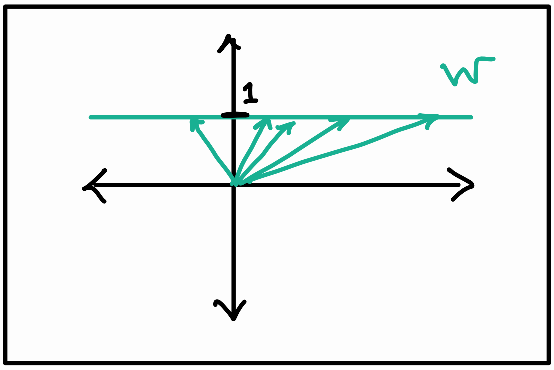
Then $\underline{u}+\underline{v} = \begin{bmatrix}a+b\\2\end{bmatrix}$, which does not fit the condition for being in $\mathcal{W}$, so $\underline{u}+\underline{v}$ has "left" $\mathcal{W}$, i.e. $\mathcal{W}$ is not closed under addition.

× 3) Closed under scalar multiplication?

Easy to see that No, $\mathcal{W}$ is not closed under s.m. either.

We could have stopped after 1 "failure", but this was for illustration.

W is not a subspace of $\mathbb{R}^2$



$V = \mathbb{R}^2$

(Ex 3) Let W be the set of all linear combinations of $\underline{x} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $\underline{y} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$,

(that means all combinations $a\underline{x} + b\underline{y}$ with $a, b \in \mathbb{R}$) (★ This means $W = \text{span}\{\underline{x}, \underline{y}\}$)

a.k.a. $W = \{ a\underline{x} + b\underline{y} : a, b \in \mathbb{R} \}$

or $= \left\{ a\begin{bmatrix} 3 \\ 1 \end{bmatrix} + b\begin{bmatrix} -1 \\ 0 \end{bmatrix} : a, b \in \mathbb{R} \right\}$

or $= \left\{ \begin{bmatrix} 3a - b \\ 1a - 0b \end{bmatrix} : a, b \in \mathbb{R} \right\}$

Is this a subspace of $\mathbb{R}^2$?

✓ 1) $\underline{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \in W$? Yes, because $\underline{0}$ is the combination $\underline{0} = 0 \cdot \underline{x} + 0 \cdot \underline{y}$.

✓ 2) Closed under addition?

Generic $\underline{u}, \underline{v}$ from W:

$\underline{u} = a\underline{x} + b\underline{y}$, $\underline{v} = c\underline{x} + d\underline{y}$

Sum $\underline{u} + \underline{v} = a\underline{x} + b\underline{y} + c\underline{x} + d\underline{y}$

$= (a+c)\underline{x} + (b+d)\underline{y}$,

this still fits the description of a linear combination of $\underline{x}$ and $\underline{y}$. So <u>Yes</u>

✓ 3) <u>Closed under scalar multip?</u>

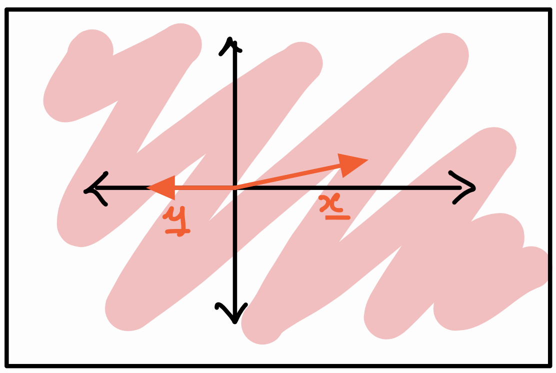
Generic $\underline{u} \in W \longrightarrow \underline{u} = a\underline{x} + b\underline{y}$

k scalar: $k\underline{u} \in W$?

$$k\underline{u} = k(a\underline{x} + b\underline{y}) = (ka)\underline{x} + (kb)\underline{y},$$

which is still in W, so <u>Yes</u>

3 "Yes" $\Longrightarrow$ W is a subspace of $\mathbb{R}^2$.



$V = \mathbb{R}^2$

W actually covers all of $\mathbb{R}^2$ (more on that later).

<u>Ex</u>: $W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 : 3x - 2y - z = 0 \right\}$

We can check that this is a subspace in two ways (really there is no difference in the two ways)

<u>Method 1</u>: Check 3 conditions.

⓪? The vector $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ is in W because $3x - 2y - z = 0$

$\downarrow$

$3(0) - 2(0) - (0) = 0$

✓ is valid.

the condition of "being in W" is that

$z = 3x - 2y$, which means that the vector's 3rd

coordinate must be $= 3(1^{st}\text{ coord}) - 2(2^{nd}\text{ coord.})$.

So, picking $\begin{cases} (1^{st}\text{ coord}) = s \\ (2^{nd}\text{ coord}) = t \end{cases}$, (free params), then our

vector must be $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} s \\ t \\ 3s - 2t \end{bmatrix}$.

CS If k is a scalar, is $\begin{bmatrix} kx \\ ky \\ kz \end{bmatrix}$ in W still?

Yes: $\begin{bmatrix} kx \\ ky \\ kz \end{bmatrix} = \begin{bmatrix} ks \\ kt \\ k(3s-2t) \end{bmatrix} = \begin{bmatrix} ks \\ kt \\ 3(ks) - 2(kt) \end{bmatrix}$

CA? Generic elements of W are

$\begin{bmatrix} a \\ b \\ 3a-2b \end{bmatrix}$ and $\begin{bmatrix} c \\ d \\ 3c-2d \end{bmatrix}$.

Is $\begin{bmatrix} a \\ b \\ 3a-2b \end{bmatrix} + \begin{bmatrix} c \\ d \\ 3c-2d \end{bmatrix}$ in W?

Yes: The sum is $= \begin{bmatrix} a+c \\ b+d \\ 3(a+c) - 2(b+d) \end{bmatrix}$

Since all 3 conditions are met <u>W is a subspace of $\mathbb{R}^3$.</u>

★ **Method 2** We saw a generic element of W is of the form $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} s \\ t \\ 3s-2t \end{bmatrix}$, where $s,t \in \mathbb{R}$, and s,t are any values (they are basically free parameters)

Every vector like that can be separated into a sum of two vectors:

$$\begin{bmatrix} s \\ t \\ 3s-2t \end{bmatrix} = s\begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} + t\begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}$$

so,
$$\left\{\begin{bmatrix} x \\ y \\ z \end{bmatrix} : 3x-2y-z=0\right\} (= W)$$

$$= \left\{\begin{bmatrix} s \\ t \\ 3s-2t \end{bmatrix} : s,t \in \mathbb{R}\right\}$$

$$= \left\{ s\begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} + t\begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix} : s,t \in \mathbb{R}\right\}$$

$$= \operatorname{span}\left\{\underbrace{\begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}}_{\text{"}\underline{u}\text{"}}, \underbrace{\begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}}_{\text{"}\underline{v}\text{"}}\right\} = \operatorname{span}\{\underline{u}, \underline{v}\}$$

Now that we know that $W = \operatorname{span}\{\underline{u}, \underline{v}\}$, it is very easy to check the subspace conditions:

defin. of span

$$W = \operatorname{span}\{\underline{u}, \underline{v}\} = \{s\underline{u} + t\underline{v} : s,t \in \mathbb{R}\}$$

$\underline{0}$? Yes, $\underline{0} \in W$ b/c $s=0, t=0 \rightarrow 0\underline{u}+0\underline{v}=\underline{0}$

(CA)? Generic elements of W are

$$\underline{x} = a\underline{u} + b\underline{v}$$
$$\underline{y} = c\underline{u} + d\underline{v}$$

is sum $\underline{x}+\underline{y}$ in W still?

Yes, b/c $\underline{x}+\underline{y} = (a+c)\underline{u} + (b+d)\underline{v}$,

which is still a linear combination of $\underline{u}, \underline{v}$

so it is in the span$\{\underline{u}, \underline{v}\} = W$.

(CS)? Given a generic $\underline{x} \in W$, we know

$\underline{x} = s\underline{u} + t\underline{v}$ for some unknown (but free) s, t.

Then $k\underline{x} = k(s\underline{u} + t\underline{v}) = (ks)\underline{u} + (kt)\underline{v}$,

still a linear comb. of $\underline{u}, \underline{v}$, so still in span$\{\underline{u}, \underline{v}\} = W$.

All 3 true, so <u>W is a subspace of $\mathbb{R}^3$.</u>

★ Note, Method 2 is "better" because of

<u>Thm</u>: If W is a span of one or more vectors from $\mathbb{R}^n$, then W is automatically a subspace of $\mathbb{R}^n$.

$\star$ [Null space : $\underline{A}$ $(n \times n)$. null($\underline{A}$) is set of sols to $\underline{A}\underline{x} = \underline{0}$ (things that $\underline{A}$ sends to $\underline{0}$)]

Ex :

$$\underbrace{\begin{bmatrix} 1 & 3 & -15 & 7 \\ 1 & 4 & -19 & 10 \\ 2 & 5 & -26 & 11 \end{bmatrix}}_{\underline{A}} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}}_{\underline{x}} = \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}}_{\underline{0}}.$$

What is Null($\underline{A}$)? Is Null($\underline{A}$) a subspace of $\mathbb{R}^4$?

Compute sols $\underline{x}$ via RREF . . .

$$\left[\begin{array}{cccc|c} \boxed{1} & 0 & -3 & -2 & 0 \\ 0 & \boxed{1} & -4 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$\underbrace{x_3 \quad x_4}_{\text{free}}$

$x_3 = s$, $x_4 = t$ free params.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3s + 2t \\ 4s - 3t \\ s \\ t \end{bmatrix} = s \begin{bmatrix} 3 \\ 4 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix}$$

so solution set, aka null($\underline{A}$) = span$\left\{ \begin{bmatrix} 3 \\ 4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix} \right\}$,

$\Longrightarrow$ (by thm) null($\underline{A}$) is a subspace of $\mathbb{R}^4$

$\star$ Null($\underline{A}$) is always a subspace (of whatever $\mathbb{R}^n$) :

null($\underline{A}$) $= \{ \underline{x} \in \mathbb{R}^n : \underline{A}\underline{x} = \underline{0} \}$ $\quad$ ($\underline{x}_{(n \times 1)}$, $\underline{0}_{(n \times 1)}$)

1) $\underline{0} \in$ Null($\underline{A}$)? Yes, $\underline{A}\underline{0} = \underline{0}$

CA? If $\underline{A}\underline{u} = \underline{0}$ $\underline{A}\underline{v} = \underline{0}$, is $\underline{A}(\underline{u}+\underline{v}) = \underline{0}$ too?

Yes : $\underline{A}(\underline{u}+\underline{v}) = \underline{A}\underline{u} + \underline{A}\underline{v} = \underline{0} + \underline{0} = \underline{0}$.

CS? If $\underline{A}\underline{u} = \underline{0}$, is $\underline{A}(k\underline{u}) = \underline{0}$?

Yes : $\underline{A}(k\underline{u}) = k(\underline{A}\underline{u}) = k\,\underline{0} = \underline{0}$.

- Sometimes term kernel (of $\underline{A}$) or ker($\underline{A}$) used.